# A METHOD OF CALCULATING THE TEMPERATURE DISTRIBUTION IN A COMPLEX STRUCTURE

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UDC 533.24.02

A mathematical model is given for the temperature distribution in a structure when the temperatures in the individual elements are described by one-dimensional heat-conduction equations.

We consider here the construction of a mathematical model for a structure; the problem of calculating the temperature distributions in the individual elements is assumed in essence to have been solved. We now consider element j in the structure. The temperature distribution in this element is described by a nonlinear heat-conduction equation:

$$\frac{\partial T_j}{\partial t} = L_j T + q_{V_j} \tag{1}$$

with the initial and boundary conditions

$$T_j|_{t=0} = T_{j_0}, \tag{2}$$

$$T_j|_{\Gamma_j} = T_{\Gamma_j}. \tag{3}$$

The simplest numerical solution to the boundary-value problem of (1)-(3) occurs [1] for a one-dimensional differential operator $L_jT$; as $L_jT$ increases in dimensions, the computing time and store volume increase rapidly, particularly on account of the nonlinearity.

When a mathematical model is being drawn up, one can incorporate the working conditions to simplify the treatment and reduce the dimensions of $L_jT$; for instance, in the case of a thin-walled shell subject to axially symmetrical heat loading, it is sufficient to examine the one-dimensional axial temperature distribution, i.e., here it is sufficient to use a one-dimensional $L_jT$ operator. Then the heat-source function $q_{V_j}$ in (1) will be the sum

$$q_{V_j} = q_{V_j}^{\text{ext}} + q_{V_j}^{\text{int}}, \tag{4}$$

where $q_{V_j}^{\text{ext}}$ is the function representing the effects of the environment on the element, together with those of a variety of heat carriers if the element is heated or cooled regeneratively, and so on, while $q_{V_j}^{\text{int}}$ is the internal heat-release function for the element.

One might quote several examples where such assumptions simplify the calculation of the temperature distribution by reducing the dimensions of $L_jT$.

The interaction between adjacent elements is incorporated by specifying the condition for continuity of the temperature and heat flux at the junction between elements:

$$T_j|_{\Gamma_n} = T_{\Gamma_n},\ n = 1,\ 2,\ \ldots,\ N_\alpha, \tag{5}$$

$$\sum_{n=1}^{N_\alpha} \lambda_n f_{\Gamma_n} \frac{\partial T_{\Gamma_n}}{\partial x_n} = 0. \tag{6}$$

Then calculation of the temperature distributions in such a structure may be formulated as follows:

S. Ordzhonikdize Moscow Aviation Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 1, pp. 140-144, July, 1975. Original article submitted February 10, 1975.

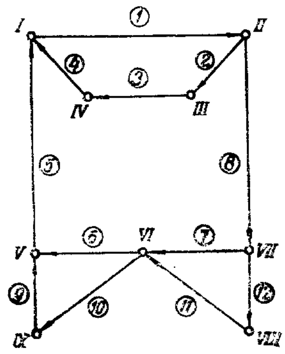


Fig. 1. The oriented graph G' (V).

one has to find a temperature distribution in a system of coupled elements with a given initial temperature of (2) if the one-dimensional equation of (1) applies to each element, while (5) and (6) are met at the boundaries of an element.

This problem can be solved in the general case by electrical simulation methods; here we consider a method that can be used with a digital computer.

We use a finite-difference approximation for the differential operators to convert the problem of (1), (2), (3), (5), and (6) to a difference problem on graphs [3,4]. We represent the system of elements as an oriented graph G' (V) (Fig. 1), on each arc of which $E_{\alpha\beta} \in G'(V)$ ($\alpha$ is the initial point of an arc and $\beta$ is the final point), where we are given the one-dimensional heat-conduction equation of (1). We introduce the net $\omega_h$ on $E_{\alpha\beta}$; the finite-difference approximation allows one to reduce the solution of (1) to that of a system of algebraic equations having a three-diagonal matrix for each arc of the graph. On the basis of the number of arc, we represent the complete system of algebraic equations as follows:

$$
\begin{gathered}
\hat{T}_{j,0}(E_{\alpha\beta}) \equiv \hat{T}_j(x_\alpha),\\
A_{j,i}(E_{\alpha\beta})\hat{T}_{j,i-1}(E_{\alpha\beta}) - C_{j,i}(E_{\alpha\beta})\hat{T}_{j,i} + B_{j,i}(E_{\alpha\beta})\hat{T}_{j,i+1}(E_{\alpha\beta}) =\\
= -F_{j,i}(E_{\alpha\beta}),\ i = 1, 2, \ldots, N_j - 1,\ N_j = N_j(E_{\alpha\beta}),\ \alpha = 1, 2, \ldots, N,\\
\hat{T}_{j,N_j}(E_{\alpha\beta}) \equiv \hat{T}_j(x_\beta),\ E_{\alpha\beta} \in G'(V),\\
\sum_{\beta \in G_\alpha^+} f_j\lambda_j[-C_{j,0}(E_{\alpha\beta})\hat{T}_{j,0}(E_{\alpha\beta}) + B_{j,0}(E_{\alpha\beta})\hat{T}_{j,1}(E_{\alpha\beta}) +\\
+ F_{j,0}(E_{\alpha\beta})] + \sum_{\beta \in G_\alpha^-} f_j\lambda_j[A_{j,N_j}(E_{\alpha\beta})\hat{T}_{j,N_j-1}(E_{\alpha\beta}) -\\
- C_{j,N_j}(E_{\alpha\beta})\hat{T}_{j,N_j}(E_{\alpha\beta}) + F_{j,N_j}(E_{\alpha\beta})] = 0.
\end{gathered} \tag{7}
$$

We derive the solution to (7) by the cyclic pivot method [5]; the desired function is then defined by

$$\hat{T}_{i+1} = \alpha_i\hat{T}_i + \beta_i + \gamma_i\hat{T}(x_\beta) \text{ on } E_{\alpha\beta}(\beta \in G_\alpha^+), \tag{8}$$

$$\hat{T}_i = \overline{\alpha}\hat{T}_{i+1} + \overline{\beta}_{i+1} + \overline{\gamma}\hat{T}(x_\beta) \text{ on } E_{\alpha\beta}(\beta \in G_\alpha^-). \tag{9}$$

The fitting coefficients $\alpha_i$, $\beta_i$, $\gamma_i$, $\overline{\alpha}_{i+1}$, $\overline{\beta}_{i+1}$, $\overline{\gamma}_{i+1}$ are selected to make the function $T_{j,i}$ satisfy system (7), while (8) and (9), respectively, with $i = N_j - 1$ and $i = 0$ become identities, i.e.,

$$
\begin{gathered}
\alpha_{i-1} = \Delta_i^{-1}A_{j,i},\\
\beta_{i-1} = \Delta_i^{-1}(F_{j,i} + B_{j,i}\beta_i),\\
\gamma_{i-1} = \Delta_i^{-1}B_{j,i}\gamma_i,\ \Delta_i = C_{j,i} - B_{j,i}\alpha_i,\\
i = N_j - 1,\ N_j - 2, \ldots, 0,\\
\overline{\alpha}_{N_j-1} = \overline{\Delta}_i^{-1}(F_{j,i} + A_{j,i}\overline{\beta}_i),\\
\overline{\gamma}_{i+1} = \overline{\Delta}_i^{-1}A_{j,i}\overline{\gamma}_i,\ \overline{\Delta}_i = C_{j,i} - A_{j,i}\overline{\alpha}_i,\\
\overline{\beta}_{i+1} - \overline{\Delta}^{-1}(A_{j,i}\overline{\beta}_i + F_{j,i}),\ i = 1, 2, \ldots, N_j,\\
\overline{\alpha}_i = \overline{\beta}_1 = 0,\ \overline{\gamma}_1 = 0.
\end{gathered} \tag{10}
$$

We put i = 0 in (8) and $i = N_j - 1$ in (9) on the basis that

$$
\begin{gathered}
\hat{T}_{N_j}(E_{\gamma\alpha}) = \hat{T}_0(E_{\alpha\beta}) = \hat{T}(x_\alpha);\\
\gamma \in G_\alpha^-,\ \beta \in G_\alpha^+,
\end{gathered} \tag{11}
$$

to get for all $\alpha = 1, 2, \ldots, N_\alpha$ [3] a system of algebraic equations for the temperatures at the vertices of the graph (at the points where the elements join):

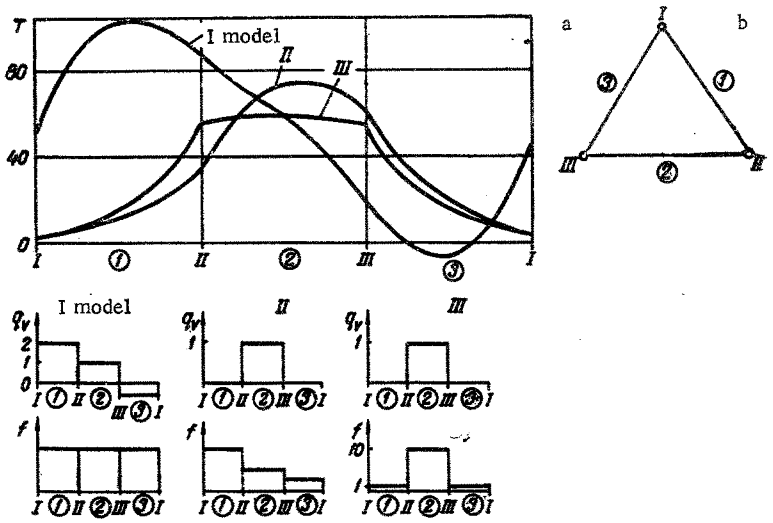


Fig. 2. (a) Temperature distribution (b) on graph at time t.

$$\hat{T}(x_\alpha) = \sum_{\beta \in G_\alpha} A_{\alpha\beta} \hat{T}(x_\beta) + \Phi_\alpha, \quad \alpha = 1, 2, \ldots, N, \tag{12}$$

where

$$A_{\alpha\beta} = D_\alpha^{-1} \begin{cases} f_j \lambda_j \Delta_0 (E_{\alpha\beta}) \gamma_{-1} (E_{\alpha\beta}), & \beta \in G_\alpha^+, \\ f_j \lambda_j \bar{\Delta}_{N_j} (E_{\alpha\beta}) \bar{\gamma}_{N_j - 1} (E_{\beta\alpha}), & \beta \in G_\alpha^-, \end{cases} \tag{13}$$

$$\Phi(\alpha) = D_\alpha^{-1} \Big[ \sum_{\beta \in G_\alpha^+} f_j \lambda_j \Delta_0 (E_{\alpha\beta}) \beta_{-1} (E_{\alpha\beta}) + \sum_{\beta \in G_\alpha^-} f_j \lambda_j \bar{\Delta}_{N_j} (E_{\alpha\beta}) \bar{\beta}_{N_j+1} (E_{\alpha\beta}) \Big], \tag{14}$$

$$D_\alpha = \sum_{\beta \in G_\alpha^+} f_j \lambda_j \Delta_0 (E_{\alpha\beta}) + \sum_{\beta \in G_\alpha^-} f_j \lambda_j \bar{\Delta}_{N_j} (E_{\beta\alpha}). \tag{15}$$

The solution to system (12) gives the temperatures at the joints, and then these can be used in solving (7) to find the temperature distribution in the structure.

As an example we give the results for the solution of (1), (2), (3), (5), and (6) as specified via the graph of Fig. 2a; all the coefficients in (1) were taken as 1. Figure 2b shows the temperature distribution on the graph for a particular instant t. The figure also shows the source functions and the cross-sectional areas of the joined elements.

To conclude, we note that thermal models for complex structures based on graphs are simple and convenient means of incorporating the interaction between elements in a fairly rigorous fashion.

## NOTATION

T, temperature; t, time; x, linear coordinate; LT, parabolic differential operator; $q_V$, source function; $q_V^{ext}$, source function for effects of ambient medium heat carrier, etc.; $q_V^{int}$, internal heat-release function; Γ, element boundary; f, contact area of parts; λ, thermal conductivity; V, set of vertices; $E_{\alpha\beta}$, arc; $G_\alpha$, set of arcs converging at vertex α; $G_\alpha^+$, set of arcs diverging from vertex α; $G_\alpha^-$, set of arcs at vertex α; $\omega_h$, net on arc $E_{\alpha\beta}$; N, set of graph vertices; $N_\alpha$, set of graph arcs converging at vertex α; $N_j$, number of nodes on arc j; $T_{j,i}$, network function at time instant k; $T_{j,i}$, network function at time instant k ≠ 0; α, β, and γ, vertex indices; j, arc index; i, node index.

## LITERATURE CITED


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